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INELASTIC COLUMN STABILITY SUBJECTED TO TIME-DEPENDENT LOAD

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Abstract—The dynamic stability of an initially straight, inelastic column subjected to a time-dependent axial load just beyond the tangent modulus load is studied. The results indicate that the straight configuration, which is stable under "dead loading", is dynamically unstable. When the load becomes constant, the column seeks a deflected, stable equilibrium configuration which depends on the nature of transitory disturbances. Furthermore, within certain bounds, there may be a continuous distribution of such configurations.

INTRODUCTION

THE stability and bifurcation phenomena in inelastic columns have been recently and extensively studied by Hill and Sewell [1]. They have shown that bifurcation of stable equilibrium configurations may occur quasi-statically at any load between the Shanley [2] and Engesser–Kármán loads. Hill has stated elsewhere [3] that when the internal work of distortion of a solid is path-dependent, consideration should be given to the effects of dynamically possible paths on stability. The purpose of this paper is to explore such effects on the stability of an inelastic column.

The word stability may have various connotations depending on the viewpoint one takes but its basic meaning has to be in a dynamic sense. The classical definition of stability by Dirichlet and Kelvin may be stated as follows. A body is in a state of stable equilibrium if, in the motion following any kind of transitory disturbance, the amplitude of the additional displacement is always vanishingly small when the disturbance itself is vanishingly small. Even within the context of this definition, different results may be obtained if not all possible disturbances have been considered. Also, at times, the definition may not be sufficiently discriminative. For instance an initially straight, inelastic column subjected to an axial load just beyond the tangent modulus load may have two equilibrium configurations: a straight form and a bent form [4]. According to the definition, it may be shown that either one of the two configurations is stable, if the load is a "dead load" and if lateral disturbances are considered only. However, available experimental results indicate that the bent form is usually preferred. Therefore, there is a difference in the degree of stability between the two configurations. It will be shown in this paper that, if the axial load is time-dependent, which is physically the case, the straight configuration is dynamically unstable. Furthermore, the instantaneous response of the column material to motion depends on the history of motion. Therefore, the course of bending motion is sensitive to the initial conditions, dynamic disturbances and the entire loading process.

Real transitory disturbances are usually irregular and random in nature. However, only the effects of those disturbances which excite the fundamental mode of deflection are analyzed. The dual response (loading and unloading) of the material gives rise to complications which make a universal description of motion difficult.

COLUMN MODEL

For convenience of analysis, a long, initially straight column of a uniform rectangular cross-section and with hinged ends is considered. Its dimensions are length l, section width b and depth $2a$ such that $2a < b$ and $2a \ll l$. The lower end of the column is stationary while the upper end may move vertically downward. At first, the column is stationary and subjected to a central vertical load P in compression beyond the stage where its behavior is purely elastic. Then the load may be increased by a total increment p applied at a constant rate S in a finite duration t_1. Meanwhile, the column may be subjected to a lateral impulsive disturbance in the form of a prescribed initial velocity distribution. The lateral motion of the column and the subsequent equilibrium configuration are to be investigated.

It is assumed that the rate of axial loading is sufficiently slow so that the internal resultant axial force may be considered constant along the entire length of the column. Furthermore, only lateral motions involving large wave lengths (compared with $2a$) are considered. Thus the effects of rotational inertia and transverse shear may be neglected. It is also assumed that the motion and deformation of the column are sufficiently small such that it is unnecessary to make a distinction between the true and nominal stresses or between the spatial and material descriptions of the motion.

For the purpose of this paper, the property of the column material may be characterized by a single tangent modulus E_t associated with incremental plastic loading and a Young's modulus E associated with elastic unloading. In other words, the axial stress rate $\dot{\sigma}$ and strain rate $\dot{\varepsilon}$ (positive in compression) are related by

$$\dot{\sigma} = E_t\dot{\varepsilon} \quad \text{for } \varepsilon = \varepsilon^* \quad \text{and } \dot{\varepsilon} > 0 \tag{1a}$$

$$\dot{\sigma} = E\dot{\varepsilon} = E\dot{\varepsilon}^e \quad \text{for } \varepsilon < \varepsilon^* \tag{1b}$$

where ε^* is the largest local compressive strain that the material has ever experienced. The total strain rate, $\dot{\varepsilon}$, may be separated into two parts, elastic strain rate $\dot{\varepsilon}^e$ and irreversible plastic strain rate $\dot{\varepsilon}^p$ such that

$$\dot{\varepsilon}^p = \begin{cases} \dot{\sigma}(1/E_t - 1/E) & \text{for } \sigma = \sigma^* \quad \text{and } \dot{\sigma} > 0 \\ 0 & \text{for } \sigma < \sigma^* \end{cases} \tag{2}$$

where σ^* corresponds to ε^*.

BASIC EQUATIONS

Let the origin of the cartesian axes of reference (x, y, z), in the undeflected configuration be at the centroid of the base, the x-axis being along the line of centroids (positive upward) and the y-axis being parallel to the depth of the sections. The respective displacement components of the centroid of a section are denoted by u, v and w. For symmetrical bending, the only deformation considered in the paper, $w = 0$. Let the axial strain at a centroid be ε_0 or $\varepsilon(x, y, t)|_{y=0} = \varepsilon_0(x, t)$. Assuming that $u \ll v$, the centroidal strain vs. displacement relationship may be written as

$$\varepsilon_0(x, t) = -\frac{\partial u(x, t)}{\partial x} - \frac{1}{2}\left[\frac{\partial v(x, t)}{\partial x}\right]^2. \tag{3}$$

Assuming the Euler–Bernoulli approach, the strain rate elsewhere may be expressed in terms of the centroidal strain rate, $\dot{\varepsilon}_0$ and the lateral velocity $\dot{v}$, or

$$\dot{\varepsilon}(x, y, t) = \dot{\varepsilon}_0(x, t) + y\frac{\partial^2 \dot{v}(x, t)}{\partial x^2}. \tag{4}$$

The bending behavior of the column depends on the position of the neutral surface, $\eta(x, t)$, defined by $\dot{\varepsilon}(x, \eta, t) = 0$ or

$$\eta(x, t) = -\frac{\dot{\varepsilon}_0}{\partial^2 \dot{v}/\partial x^2}. \tag{5}$$

The position of the neutral surface in turn depends on the axial equilibrium condition

$$b\int_{-a}^{a} \dot{\sigma}\,\mathrm{d}y = \begin{cases} S & \text{for } 0 \le t \le t_1 \\ 0 & \text{for } \quad t > t_1 \end{cases}. \tag{6}$$

The rate of change of internal bending moment, $\dot{M}_i$, about a centroidal axis of a section is given by

$$\begin{aligned} \dot{M}_i &= -b\int_{-a}^{a} \dot{\sigma} y\,\mathrm{d}y \\ &= -E^*(\eta) I\frac{\partial^2 \dot{v}}{\partial x^2} \end{aligned} \tag{7}$$

where $I = \frac{2}{3}ba^3$ and the effective modulus E^* depends on η. The active moment consists of two parts: the moment produced by the axial load and the moment produced by the transverse inertia forces. By using D'Alembert's principle and by balancing the internal and active moments, the following equation of motion is obtained:

$$\begin{aligned} &\frac{\partial^2}{\partial x^2}\left\{I\int_0^t E^*[\eta(x, \tau)]\frac{\partial^2 \dot{v}(x, \tau)}{\partial x^2}\,\mathrm{d}\tau\right\} \\ &\qquad + (P + St)\frac{\partial^2 v(x, t)}{\partial x^2} + m\frac{\partial^2 v(x, t)}{\partial t^2} = 0 \end{aligned} \tag{8}$$

where m is the mass per unit length of the column. Equation (8) together with equation (6) determine the functions $u(x, t)$ and $v(x, t)$. Equation (8) indicates that the lateral motion and the material response are inter-related. The solution of equation (8) depends on the initial conditions and the history of loading process. In general, a piecewise approach may be necessary for solving equation (8), because the derivatives of the effective modulus with respect to time or space may be piecewise continuous.

CASE OF CONSTANT EFFECTIVE MODULUS

When the initial lateral disturbance is relatively small, the situation $\dot{\varepsilon} \ge 0$ and $\varepsilon = \varepsilon^*$ may occur everywhere in the column within the time interval $0 \le t \le t_1$. In other words, by equations (1, 4, and 6),

$$\dot{\varepsilon}_0 = \frac{S}{2abE_t}$$

and

$$\frac{\partial^2 \dot{v}}{\partial x^2} < \frac{S}{2a^2 b E_t} \quad \text{for } 0 \le t \le t_1. \tag{9}$$

In this case, the effective modulus $E^* = E_t$ is a constant and equation (8) admits a simple solution corresponding to the fundamental mode of motion of the form

$$v = f(t) \sin \frac{\pi x}{l} \text{ for } 0 \le t \le t_1 \tag{10}$$

which satisfies the boundary conditions $v = \partial^2 v/\partial x^2 = 0$ at $x = 0, l$. Substituting v from equation (10) into equation (8), the following equation is obtained for determining the function $f(t)$:

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + \frac{\pi^2}{ml^2}\left(\frac{\pi^2 E_t I}{l^2} - P\right) f - \frac{\pi^2 S t}{ml^2} f = 0. \tag{11}$$

Using the following factors:

$$\omega^2 = \frac{\pi^2}{ml^2} P_t \left(1 - \frac{P}{P_t}\right) \tag{12}$$

and

$$\lambda = \frac{\pi^2 S}{ml^2} \tag{13}$$

where, the tangent modulus load

$$P_t = \frac{\pi^2 E_t I}{l^2}, \tag{14}$$

equation (11) may be written as

$$\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} + (\omega^2 - \lambda t) f = 0. \tag{15}$$

This equation may be solved by assuming a series solution of the form

$$f = \sum_{n=0}^{\infty} a_n t^{n+k}. \tag{16}$$

The general solution of equation (15) is

$$\begin{aligned} f = A\Bigg\{ & \frac{1}{\omega}\left[\omega t - \frac{(\omega t)^3}{3!} + \frac{(\omega t)^5}{5!} - \frac{(\omega t)^7}{7!} + \dots\right] \\ & + \left[\frac{2\lambda t^4}{4!} + \frac{10\lambda^2 t^7}{7!} + \frac{80\lambda^3 t^{10}}{10!} + \dots\right] \\ & - \omega^2\left[\frac{6\lambda t^6}{6!} + \frac{52\lambda^2 t^9}{9!} + \dots\right] \end{aligned}$$

$$
\begin{aligned}
&+\omega^4\left[\frac{12\lambda t^8}{8!}+\frac{160\lambda^2 t^{11}}{11!}+\ldots\right] \\
&-\omega^6\left[\frac{20\lambda t^{10}}{10!}+\ldots\right]+\ldots\Bigg\} \\
&+B\Bigg\{\left[1-\frac{(\omega t)^2}{2!}+\frac{(\omega t)^4}{4!}-\frac{(\omega t)^6}{6!}+\ldots\right] \\
&+\left[\frac{\lambda t^3}{3!}+\frac{4\lambda^2 t^6}{6!}+\frac{28\lambda^3 t^9}{9!}+\ldots\right] \\
&-\omega^2\left[\frac{4\lambda t^5}{5!}+\frac{28\lambda^2 t^8}{8!}+\ldots\right] \\
&+\omega^4\left[\frac{9\lambda t^7}{7!}+\frac{100\lambda^2 t^{10}}{10!}+\ldots\right] \\
&-\omega^6\left[\frac{16\lambda t^9}{9!}+\ldots\right]+\ldots\Bigg\}
\end{aligned}
\tag{17}
$$

where the constant A indicates the amplitude of the initial velocity distribution and the constant B represents the magnitude of the initial geometrical deviation from a straight column. In the subsequent analysis, A is used only to indicate the intensity of the lateral disturbance. It is to be noted that, by a ratio test, the series in equation (17) is absolutely convergent for $(\omega t)^2 < \infty$ and $\lambda t^3 < \infty$. An examination of equation (17) shows that the lateral displacement grows monotonically with time for the case $\omega^2 \leq 0$ or $P/P_t \geq 1$. This means that the straight configuration of the column under a load larger than or equal to the tangent modulus load is intrinsically unstable. It is to be noted that equations (10 and 17) are valid as long as inequality (9) holds. As time progresses, inequality (9) will be eventually violated, and unloading may develop in some parts of the column. Then the motion may follow some other course governed by equation (8).

It would be of interest to examine the case $P/P_t < 1$. For a better picture of the behavior, the solution by equation (17) may be expressed in the following form

$$
\begin{aligned}
f = {}& A'\left[\sin\omega t+\frac{\lambda}{4}\left(\frac{t}{\omega^2}\sin\omega t-\frac{t^2}{\omega^2}\cos\omega t\right)+\ldots\right] \\
&+B'\left[\cos\omega t+\frac{\lambda}{4}\left(\frac{t}{\omega^2}\cos\omega t+\frac{t^2}{\omega}\sin\omega t\right)+\ldots\right].
\end{aligned}
\tag{18}
$$

Equation (18) shows that the column may develop an oscillatory motion in the neighborhood of the straight configuration with its amplitude gradually increasing with time. The rate of increase depends on the factor λ/ω. When $P/P_t \to 1$, $\omega \to 0$ and the motion diverges.

When $t = t_1$, the amplitude function f, by equation (17), may have a value of f_1 and its first derivative with respect to time may have a value A_1. By inequality (9), it may be shown that

$$
f_1 = \frac{Cap}{3P_t} \quad \text{for } 0 < C < 1 \tag{19}
$$

where C is simply a proportional factor. By examining equation (17), it is obvious that $A_1 > A$ and that A_1, f_1 and A have the same sign for the case $P/P_t \geq 1$, but not necessarily so for the case $P/P_t < 1$. The deflection and momentum attained by the column at $t = t_1$ will induce the following motion. Let the lateral deflection of the column be

$$v(x, t) = f_1 \sin\frac{\pi x}{l} + v'(x, t) \quad \text{for } t_1 < t < t_3 \tag{20}$$

and when

$$t = t_1, \qquad v' = 0, \qquad \frac{\partial v}{\partial t} = \frac{\partial v'}{\partial t} = A_1 \sin\frac{\pi x}{l} \tag{21}$$

where t_3 is the time when the direction of motion reverses. It may be shown by equation (6) that unloading occurs in part of the column. However the position of the neutral surface is constant and is given by

$$\eta_1 = \frac{\sqrt{E}-\sqrt{E_t}}{\sqrt{E}+\sqrt{E_t}}a \quad \text{for } t_1 < t < t_3. \tag{22}$$

By equation (7), the internal bending moment is found

$$M_i = \frac{Cap}{3}\sin\frac{\pi x}{l} - E_r I\frac{\partial^2 v'}{\partial x^2} \quad \text{for } t_1 < t < t_3 \tag{23}$$

where $E_r = 4EE_t/(\sqrt{E}+\sqrt{E_t})^2$ is the so-called reduced modulus. By equation (8), the equation of motion for the time interval may be written as

$$E_r I\frac{\partial^4 v'}{\partial x^4} + (P+p)\frac{\partial^2 v'}{\partial x^2} - \frac{\pi^2}{l^2}\left(\frac{P+p}{P_t} - 1\right)\frac{Cap}{3}\sin\frac{\pi x}{l} + m\frac{\partial^2 v'}{\partial t^2} = 0. \tag{24}$$

Using the initial conditions given by equation (21), the solution of equation (24) is found to be

$$v' = \left\{\frac{A_1}{\omega'}\sin\omega'(t-t_1) + \frac{\pi^2}{m\omega'^2 l^2}\left(\frac{P+p}{P_t} - 1\right)\frac{Cap}{3}[1-\cos\omega'(t-t_1)]\right\} \sin\frac{\pi x}{l} \quad \text{for } t_1 < t < t_3 \tag{25}$$

where

$$\omega'^2 = \frac{\pi^2 P_r}{ml^2}\left(1 - \frac{P+p}{P_r}\right) \tag{26}$$

and

$$P_r = \frac{\pi^2 E_r I}{l^2} \tag{27}$$

is the Engesser–Kármán load.

The time t_3 may now be determined by the condition that the forward velocity is reduced to zero at t_3 or

$$t_3 = t_1 + \frac{1}{\omega'} \arctan \left[\frac{-A_1}{\frac{\pi^2}{m\omega' l^2}\left(\frac{P+p}{P_t} - 1\right)\frac{Cap}{3}} \right]. \tag{28}$$

The corresponding displacement function is found to be

$$\begin{aligned} v_3 = v(x, t_3) = &\left\{ \frac{1}{\omega'} \left[\sqrt{\left(A_1^2 + \frac{\pi^4}{m^2 \omega'^2 l^4} \left(\frac{P+p}{P_t} - 1 \right)^2 \left(\frac{Cap}{3} \right)^2 \right)} \right. \right. \\ &\left. \left. + \frac{\pi^2}{m\omega' l^2} \left(\frac{P+p}{P_t} - 1 \right) \frac{Cap}{3} \right] + \frac{Cap}{3P_t} \right\} \sin \frac{\pi x}{l}. \end{aligned} \tag{29}$$

Equations (25–29) indicate the following interesting facts. First, the angular frequency of the oscillation is increasing because $E_r > E_t$ and $\omega' > \omega$. Second, if $(P+p)/P_r \to 1$, the motion diverges. This means that the column is unstable when it is subjected to a load equal to the reduced modulus load, for constant or time dependent loads. Third, the maximum lateral displacement and the corresponding time t_3 of the case $P/P_t < 1$ are less than that of the case $P/P_t \geq 1$.

At the time t_3, the column reverses its direction of motion. In the subsequent motion, it may be shown that the position of the neutral surface is no longer constant with respect to time or space and the stress histories become more complex. Analytical determination of the motion beyond time t_3 becomes difficult. It would be possible to employ a digital computer to determine the subsequent motion if necessary. However, a qualitative description may be sufficient for the present purpose.

During the time interval $0 < t < t_3$, most of the initial kinetic energy has been expended in performing irreversible plastic work on the column material. Meanwhile, the material has been work-hardened and the effective modulus everywhere quickly approaches the elastic modulus in the subsequent cycles of motion. The angular frequency increases while the amplitude of the oscillation diminishes with each cycle of motion. If no additional axial loading or lateral disturbance occurs, the column may come to rest finally. The final equilibrium configuration cannot be determined exactly without analyzing the total motion. However, the equilibrium configuration may be within certain geometrical bounds which may be determined by evaluating the work and energy.

WORK AND ENERGY

Consider an arbitrary inelastic solid of instantaneous volume V, surface area Ω, and mass density ρ. The body may be currently subjected to body forces F_i, surface tractions T_i, and internal stresses σ_{ij}. A scalar product of each side of the equation of motion

$$\sigma_{ji,j} + \rho F_i = \rho \ddot{u}_i \tag{30}$$

with the actual velocity $\dot{u}_i$ of an element of the body may be formed. The repeated subscript j implies the conventional summation in a cartesian coordinate system. By integrating the scalar products over V, the following work-energy rate equation may be obtained:

$$\dot{U}+\dot{D}-\dot{W}+\dot{K} = 0. \tag{31}$$

Where $\dot{U}$ is the rate of accumulating elastic strain energy or

$$\dot{U} = \int_V \sigma_{ij}\dot{\varepsilon}^e_{ij}\,\mathrm{d}V; \tag{32}$$

$\dot{D}$ is the rate of dissipating internal work or

$$\dot{D} = \int_V \sigma_{ij}\dot{\varepsilon}^p_{ij}\,\mathrm{d}V; \tag{33}$$

$\dot{W}$ is rate of work performed by the external forces or

$$\dot{W} = \int_V \rho F_i\dot{u}_i\mathrm{d}V+\int_\Omega T_i\dot{u}_i\,\mathrm{d}\Omega; \tag{34}$$

and $\dot{K}$ is the rate of accumulating kinetic energy or

$$\dot{K} = \frac{\mathrm{d}}{\mathrm{d}t}\int_V \tfrac{1}{2}\rho\dot{u}_i\dot{u}_i\,\mathrm{d}V. \tag{35}$$

In equations (32 and 33), $\dot{\varepsilon}^e_{ij}$ and $\dot{\varepsilon}^p_{ij}$ are the elastic and plastic part of the strain rate $\dot{\varepsilon}_{ij}$, respectively, or

$$\dot{\varepsilon}_{ij} = \dot{\varepsilon}^e_{ij}+\dot{\varepsilon}^p_{ij} = \tfrac{1}{2}(\dot{u}_{i,j}+\dot{u}_{j,i}). \tag{36}$$

The rate equation (31) is considered because of the dual characteristics of the stress–strain relationship. However, equation (31) may be integrated with respect to time to obtain a relationship between the work and energy differences. It is to be noted that the rate of dissipating internal work is always positive or irreversible and that the elastic strain energy is only partially reversible when internal constraints exist. Equation (31) may now be used to evaluate the energy balance in the column problem.

Referring to the motion of the column in the time interval $0 \le t \le t_1$ described in the last section, the elastic strain energy, dissipative and external work differences according to equations (32–34) are found to be

$$U_1-U_0 = \frac{pl}{2abE}\left(P+\frac{p}{2}+\frac{C^2p}{12}\right) \tag{37}$$

$$D_1-D_0 = \left(\frac{1}{E_t}-\frac{1}{E}\right)\frac{pl}{2ab}\left(P+\frac{p}{2}+\frac{C^2p}{12}\right) \tag{38}$$

and

$$W_1-W_0 = \frac{pl}{2abE_t}\left[P+\frac{p}{2}+\frac{C^2p}{12P_t}(P+\tfrac{2}{3}p)\right] \tag{39}$$

where the subscripts 0 and 1 refer to the time $t = 0$ and $t = t_1$, respectively. In calculating $W_1 - W_0$, the axial displacement rate $\dot{u}$ at $x = l$ by equation (3) is used. In view of equation

(31), the kinetic energy difference is found to be

$$K_1 - K_0 = \frac{C^2p^2l}{24abE_t}\left(\frac{P}{P_t} - 1 + \frac{2}{3}\frac{p}{P_t}\right). \tag{40}$$

An examination of equation (40) shows that there is a gain in the kinetic energy at t_1 compared to that at $t = 0$, if

$$P > P_t\left(1 - \frac{2}{3}\frac{p}{P_t}\right), \tag{41}$$

and there is a loss, if

$$P < P_t\left(1 - \frac{2}{3}\frac{p}{P_t}\right). \tag{42}$$

If the kinetic energy gain or loss is used as a criterion for the instability or stability, then the above evaluation again indicates that the initial straight configuration is stable when $P < P_t$ (with $p \to 0$) and unstable when $P > P_t$.

An examination of equation (25) shows a time-independent term. It represents a configuration, to be designated by $v_2 = v(x, t_2)$, occurring at a time t_2 when the total kinetic energy $K = K_2$ is a maximum. It may also be shown that v_2 is an equilibrium configuration in the sense that the internal and external forces are in equilibrium and that the inertial forces are zero because $\partial^2 v/\partial t^2 = 0$ at $t = t_2$. Furthermore, when K_2 is a maximum, by equation (31), the sum $(U_2 + D_2 - W_2)$ is a minimum. Because $\dot{D}$ is always positive, the potential energy of the system $(U-W)$ can be a minimum only at a time $t \geq t_2$. At time t_3, $K_3 = 0$ and $(U_3 + D_3 - W_3)$ is a maximum. However, $(U_3 - W_3)$ may have any relative value. If $(U_3 - W_3)$ is a minimum, v_3 is a static equilibrium configuration. If $(U_3 - W_3)$ is not a minimum, the direction of motion reverses and the column, under the prescribed conditions, eventually seeks a final equilibrium configuration of a lower potential energy of a configuration $v_\infty(x, \infty)$ within the bounds

$$v_2(x, t_2) \geq v_\infty(x, \infty) \geq v_3(x, t_3). \tag{43}$$

In general, when E_t approaches E, v_∞ approaches v_2; when E_t approaches zero, v_∞ approaches v_3.

CASE OF VARYING EFFECTIVE MODULUS

When the initial lateral disturbance is sufficiently strong, inequality (9) may be violated at the very beginning of the motion. In that case, the effective modulus is no longer a constant and an exact analytical solution of equation (8) is not readily available. To seek an approximate solution of this phase of the problem, the energy criterion is employed and the following suggestion is made.

Assume that the time duration $0 < t < t_1$ is sufficiently short such that the lateral velocity distribution remains a half-sine wave in shape for the entire duration if it is initially so. In other words, let the average velocity distribution be

$$\dot{v} = \frac{CaS}{3P_t}\sin\frac{\pi x}{l} \quad \text{for } 0 < t < t_1 \tag{44}$$

except that, for unloading to occur, the lateral disturbance must be such that $C > 1$. Knowing the velocity distribution, the average axial strain rate $\dot{\varepsilon}_0$ may be determined by equation (1, 4–6) or

$$\dot{\varepsilon}_0 = \frac{S}{2abE_t} \quad \text{for } 0 < x < hl \tag{45}$$

and

$$\dot{\varepsilon}_0 = \frac{S}{2abE_t}\left\{\frac{1+\alpha}{1-\alpha}C\sin\frac{\pi x}{l} - \sqrt{\left[\frac{4C\alpha}{1-\alpha}\sin\frac{\pi x}{l}\left(\frac{C}{1-\alpha}\sin\frac{\pi x}{l}-1\right)\right]}\right\} \text{for } hl < x < \frac{l}{2} \tag{46}$$

where

$$h = \frac{1}{\pi}\sin^{-1}\frac{1}{C} \tag{47}$$

and

$$\alpha = \frac{E_t}{E}. \tag{48}$$

The strain rate distribution is symmetrical with respect to the $x = l/2$ line. The total work performed by the axial load during the time interval may be determined by equations (3, 34, 44–46) or

$$W_1 - W_0 = \frac{pl}{2abE_t}\left\{(2P+p)\left(h + \frac{1+\alpha}{1-\alpha}\frac{\sqrt{(C^2-1)}}{\pi} - \beta_1\right) + \frac{C^2 p}{12}\left(\frac{P}{P_t} + \frac{2}{3}\frac{p}{P_t}\right)\right\} \tag{49}$$

where

$$\beta_1 = \frac{1}{\pi}\int_h^{\pi/2}\sqrt{\left[\frac{4C\alpha}{1-\alpha}\sin\theta\left(\frac{C}{1-\alpha}\sin\theta - 1\right)\right]}\,d\theta. \tag{50}$$

By equations (1, 4, 5, 32 and 33), the total energy stored and dissipated in the material in the period is found to be

$$\begin{aligned}(U_1+D_1)-(U_0+D_0) = \frac{pl}{2abE_t}\Bigg\{&2P\left(h + \frac{1+\alpha}{1-\alpha}\frac{\sqrt{(C^2-1)}}{\pi} - \beta_1\right) + p\Bigg\{\left(1+\frac{C^2}{6}\right)h - \frac{\sqrt{(C^2-1)}}{6\pi}\\ &+ \frac{2(1+\alpha)}{3(1-\alpha)^2}\left[\left(\tfrac{1}{2}-h\right)C^2 + \frac{\sqrt{(C^2-1)}}{\pi}\right] - \frac{2}{3(1-\alpha)}(\alpha\beta_1 + 2C\beta_2)\Bigg\}\Bigg\}\end{aligned} \tag{51}$$

where

$$\beta_2 = \frac{1}{\pi}\int_h^{\pi/2}\sin\theta\sqrt{\left[\frac{4C\alpha}{1-\alpha}\sin\theta\left(\frac{C}{1-\alpha}\sin\theta - 1\right)\right]}\,d\theta. \tag{52}$$

According to equation (31), the kinetic energy gain or loss during the period is equal to the difference between the external and internal work given by equations (49 and 51) or

$$\begin{aligned}K_1 - K_0 = \frac{p^2 l}{2abE_t}\Bigg\{&\frac{\sqrt{(C^2-1)}}{\pi}\,\frac{3-6\alpha-5\alpha^2}{6(1-\alpha)^2}\\ &+ \frac{C^2}{12}\left[\frac{P}{P_t} - 2h + \frac{2}{3}\frac{p}{P_t} - \frac{8(1+\alpha)}{(1-\alpha)^2}\left(\tfrac{1}{2}-h\right)\right]\\ &+ \frac{1}{3(1-\alpha)}[(5\alpha-3)\beta_1 + 4C\beta_2]\Bigg\}\end{aligned} \tag{53}$$

Equation (53) indicates that the energy gain or loss depends on the four dimensionless factors C, α, P/P_t and p/P_t (h is a function of C). By evaluating the various terms of equation (53), a few interesting facts are found. When $C = 1$, equation (53) may be reduced to equation (40). When $[P/P_t + \frac{2}{3}(p/P_t)] < 1$, there will be a kinetic energy loss for any value of $C \geq 1$. For the case $P/P_t = 1$, $p/P_t \to 0$ and $\alpha \to 1$, it may be shown that

$$\beta_1 \to \frac{2}{\pi}\frac{\sqrt{(C^2-1)}}{1-\alpha}, \tag{54}$$

$$\beta_2 \to \left[\frac{C}{(1-\alpha)^2}(\tfrac{1}{2}-h) + \frac{\sqrt{(C^2-1)}}{\pi C(1-\alpha)}\right] \tag{55}$$

and that there will be a kinetic energy gain for any $C > 1$. In this case the load approaches the elastic Euler buckling load. Thus any additional axial load and lateral disturbance would cause the motion to diverge.

For other values of P/P_t, p/P_t and α, a numerical evaluation of equation (53) will lead to a critical value of C, say C_c, beyond which there will be a kinetic energy loss. For example when $P/P_t = 1$, $p/P_t \to 0$; $\alpha = 0{\cdot}2$, $K_1 - K_0 < 0$ for any $C > 1{\cdot}01$. In fact, as long as $\alpha < 1$ and $P/P_t = 1$, a critical value of C_c exists such that $K_1 - K_0 < 0$ for any $C > C_c$. In other words, when the axial load is in the vicinity of the tangent modulus load and when the initial disturbance causes a portion of the column to be in the state of unloading, the initial kinetic energy will be dissipated and the column eventually will seek a deflected equilibrium configuration.

Although the velocity field posed by equations (44–46) is dynamically and kinematically admissible, it may not be that of a critical dynamic "path". A critical dynamic path of an inelastic body may be defined as follows. Assume that the body is subjected to prescribed, time-dependent surface tractions and displacements. In the course of applying the dynamic tractions, the body may be subjected to arbitrary, transient external disturbances. A critical dynamic path is the motion following a certain combination of external transient disturbances such that the rate of gaining kinetic energy is relatively a maximum or the rate of losing kinetic energy is relatively a minimum. The difficulty of determining such a critical path is not only in solving an equation of motion such as equation (8) but more so in finding that critical combination of external transient disturbances. These difficulties may be more thoroughly resolved in the future. The plausible results obtained by the velocity field given by equations (44–46) indicate that the velocity field is very close to that of a critical path, at least for cases having a value of C slightly larger than unity.

CONCLUDING REMARKS

It is clear from the foregoing analysis that when the internal work of distortion of a solid is path-dependent, its stability depends on the loading process and the characteristics of external dynamic disturbances. For the case of an initially straight, inelastic column subjected to an axial load slightly beyond the tangent modulus load, the straight configuration, which is stable by a quasi-static analysis, is dynamically unstable. After external disturbance has diminished, the column seeks a deflected, stable equilibrium configuration. It has been shown that, within certain bounds, there may be a continuous distribution of such configurations. Thus consideration of dynamically possible loading paths may lead to more complete knowledge of the behavior of non-linear, non-conservative solids which may not be obtained by quasi-static considerations alone.

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Résumé—La stabilité dynamique d'une colonne initiallement droite, non élastique soumise à une charge axiale dépendant du temps juste au delà de la charge du module tangentielle est étudiée. Les résultats indiquent que la configuration droite, qui est stable sous une charge morte, est dynamiquement instable. Lorsque la charge devient constante, la colonne recherche une configuration à équilibre stable, dévié qui dépend de la nature des perturbations transitoires. En outre, à l'intérieur de certaines limites, il peut y avoir une distribution continue de telles configurations.

Zusammenfassung—Die dynamische Stabilität eines anfänglich geraden unelastischen stabes der einer zeitabhängigen Axialbelastung die über der Tangentenmodulbelastung unterworfen ist, wird untersücht. Die Resultate zeigen eine gerade Konfiguration die unter "toter Belastung" stäbil ist aber dynamisch unstabil. Wenn die Belastung konstant wird, neigt der stab zu einer stabilen Gleichgewichtskonfiguration, die von der vorübergehenden Störung abhängig ist. Ferner können, im Rahmen gewisser Grenzen, fortlaufende Verteilungen dieser Konfigurationen auftreten.

Абстракт—Исследуется динамическая устойчивостъ началъно прямолинейной, неупругой колонны, подверженной осевой нагрузке, зависящей от времени и незначителъно превышающей касателъномо-дулъную критическую силу. Резулътаты показуют, что прямолинейная форма, которая является устойчивой при "мертвой нагрузке", оказывается динамически неустойчивой. Когда нагрузка становится постоянной, колонна приобретает изогнутую форму устойчивого равновесия, которая зависит от природы временных возмущений. Далее, при некоторых ограничениях, может существоватъ непрерывное распределение таких форм.